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A perturbation method for the radiation of short surface waves in three dimensions

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SUMMARY

This paper is an extension of an earlier paper ([1]) on the subject of the radiation of waves by a heaving circular cylinder in two-dimensions. In this paper the same ideas of boundary layer expansions are employed and extended to the general three-dimensional case. The ray method is used to extend the solution from a locally two-dimensional solution to the general far field solution. In the case that caustics occur, a uniform method is described to avoid solutions which become locally non-valid.

1. Introduction

The work presented in this paper is an extension of the two-dimensional treatment of a similar problem, see Hermans [1]. In that paper a perturbation method for the radiation of short surface waves is developed for a heaving circular cylinder. A two-dimensional theory is applied for the determination of the local solution. The method for the determination of the regular solution is easily translated into the three-dimensional problem, because the theory of the Green's functions is used in [1]. For the determination of the wave solution an approximation of the fluid velocity near the heaving object is needed. In [1] it is shown that for the determination of this velocity we have to solve a double body problem. Generally it is possible to write down a solution of this problem with the help of a Green's function or to write down an integral equation for the potential function on the object. Both methods are equivalent. A lot of work must be done either to construct the Green's function or to solve the integral equation analytically. In most cases it is not possible to do so. Therefore it seems preferable to use numerical methods to construct the correct fluid velocity near the object. There is a large amount of literature and programs available for this particular problem. This remark seems to be not obvious at this moment because a short wavelength is involved, however, the surface condition for the regular solution is a very simple one and the short wavelength is not a parameter for this problem.

Once this fluid velocity is known, it is possible to find the wave contribution by means of an asymptotic perturbation technique. It turns out that we have to split up the local region (near the free surface) into an inner and an outer region. It becomes obvious that the appropriate methods to tackle the local problem are the methods described in the ray theory developed in geometrical optics (Keller [3]). Some restrictions will be made to the geometry of the radiating object. The tangent plane to the object at the free surface is vertical; this is a rather serious restriction. Furthermore the intersection of the hull of the object with the free surface is supposed to be a smooth curve. The reader who is familiar with the ray method, will notice that the latter restriction is not a serious one. The first restriction can be removed as well, however, it is more convenient to treat that problem together with horizontal motions of the object. No restrictions are made with respect to convexity of the intersection curve. If the intersection curve is not convex, a caustic will be generated and the ray method leads to a singular solution near that caustic line. This problem can be overcome by means of a boundary layer solution near the caustic. However, in this paper the theory of Ludwig [4] is applied. This theory immediately leads to a uniform expansion which is valid near the caustic as well.

The validity of the solutions will not be questioned. For most solutions found by means of the ray method no proofs have been given. A comparison with known solutions gives an indication that the method may be applied generally and leads to correct results. In [1] a comparison is made with results obtained by Ursell [6]. It turned out that the correct results are found.

In the present paper the final calculations are carried out for a semi-submerged heaving sphere. In this case an explicit solution for the fluid velocity near the sphere is known and therefore an explicit result for the wave solution can be constructed. From the correctness of these results it is expected that the correct results for the general case are obtained.

For simplicity's sake infinite water depths are considered. This is a minor simplification. As finite depth makes the double body problem more complicated it is left out of consideration. The wave phenomenon is a local effect if the wavelength is short. Therefore the treatment of the wave part is not influenced by this simplification.

2. Formulation of the problem

In the introduction we already mentioned that we consider the depth of the water to be infinite. It will be clear from the following treatment that this is a minor simplification. The viscosity and compressibility are negligible and the wave amplitude is small, i.e. the heave amplitude which will be prescribed has to be small. Therefore linearized equations apply to this problem.

The (x, y, z) coordinate system is chosen such that the positive y-axis is vertically upwards and the plane $y=0$ coincides with the free surface in its undisturbed position, see fig. 1. The surface of the heaving object is given as $f(x, y, z)=0$ and at $y=0$ the normal to this surface lies in the plane $y=0$, therefore, $f_y(x, 0, z)=0$. The principal radii of curvature of the surface of the object are finite and non-zero if y tends to zero. The latter condition turns out to be important for the stretching procedure. If this condition is not satisfied a different stretching has to be carried out.

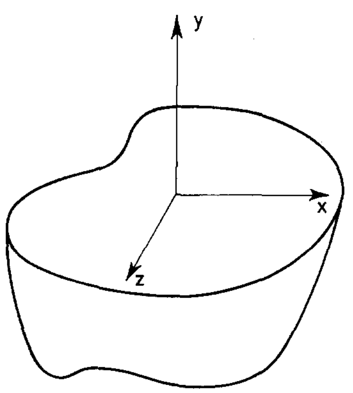

Figure 1

The heaving motion will be prescribed as follows: any point of the object performs a vertical motion

$$y = \text{Re}\left\{\frac{U}{i\omega}\, e^{-i\omega t}\right\} + y^* ,$$

where y^* denotes the undisturbed y-coordinate of that point. Therefore all time-dependent quantities in the problem are assumed to depend harmonically on t with a period $2\pi/\omega$.

The irrotational motion of an ideal fluid can be expressed in terms of a velocity potential $\Phi_{(t)}$ which for the three-dimensional time-periodic case has the representation

$$\Phi_{(t)}(x, y, z, t) = \text{Re}\{\Phi(x, y, z)\, e^{-i\omega t}\} . \tag{2.1}$$

It is assumed that this potential exists and that the free surface condition may be linearized. We notice that this is a linearization with respect to the small amplitude. Then the potential Φ satisfies:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{in the fluid (say } f(x, y, z) > 0,\ y < 0) \tag{2.2}$$

$$\frac{\partial \Phi}{\partial y} - k\Phi = 0 \qquad \text{on } y = 0,\ f(x, 0, z) > 0 \tag{2.3}$$

$$\frac{\partial \Phi}{\partial n} = \boldsymbol{U} \cdot \nabla f \qquad \text{on } f(x, y, z) = 0 \tag{2.4}$$

where $\boldsymbol{U} = (0, U, 0)$, $k = \omega^2/g$ and g denotes the acceleration of gravity. At large distance from the object the radiation condition has to be taken into account. This condition will be specified later. It turns out to be more convenient to impose an "outgoing condition" locally, which is a generally accepted technique in ray optics and it implies the usual radiation condition.

We suppose that all length-parameters, which play a role in the solution of the problem are of the same order of magnitude (L). From now on we take the unit of length equal to L and suppose all quantities to be dimensionless. Short wavelengths ($k \gg 1$) are considered. Looking at equations (2.2)–(2.4) it is immediately clear that the wave number k plays an important role in the free surface condition (2.3) and is of minor importance in (2.2) and (2.4). For $k \gg 1$ the free surface condition reduces to $\Phi = 0$ on $y = 0$. With this reduced free surface condition no wave contribution can be found. However, it leads to the correct fluid velocity near the object and away from the free surface. Therefore it yields the correct regular solution and the local solution (near $y = 0$) has to be matched with this regular solution. Obviously the local solution represents the waves.

First the regular solution will be found. It is assumed that $\Phi(x, y, z; k)$ can be represented as a regular asymptotic power series with respect to k^{-1} as follows:

$$\Phi(x, y, z; k) = \sum_{i=0}^{N} k^{-i} \psi_i(x, y, z) + o(k^{-N}), \quad \text{as } k \to \infty . \tag{2.5}$$

As mentioned in the introduction there are several ways to construct ψ_i. The use of the Green's function is explained here. After substitution of (2.5) into (2.2)–(2.4) and comparison of powers of k the following set of recurrence equations is obtained

$$\frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial y^2} + \frac{\partial^2 \psi_i}{\partial z^2} = 0 \quad \text{in the fluid} \tag{2.6}$$

$$\psi_i = \frac{\partial \psi_{i-1}}{\partial y} \ \text{on } y = 0 \ \text{ and } \ f(z, 0, z) > 0 \tag{2.7}$$

$$\frac{\partial \psi_i}{\partial n} = \delta_i^0 \, \boldsymbol{U} \cdot \nabla f \ \text{on } f(x, y, z) = 0 \tag{2.8}$$

where $\delta_i^0 = \begin{cases} 1 \text{ if } i = 0 \\ 0 \text{ if } i \neq 0 . \end{cases}$

Because of the reduction of the free surface condition to (2.7), which is of a lower order than (2.3), no wave solution exists. Therefore no radiation condition exists for the function ψ_i. Hence an other condition is imposed at infinity. The appropriate condition is to require that $\psi_i(R)$ tends to zero if R goes to infinity, where

$$R = (x^2 + y^2 + z^2)^{\frac{1}{2}} . \tag{2.9}$$

The above set of problems for ψ_i will be solved by the introduction of a Green's function for the region occupied by the fluid. The system is self-adjoint. Thus we define the Green's function $G(\boldsymbol{x}, \boldsymbol{\xi})$ as follows:

$$G_{xx}+G_{yy}+G_{zz}=0 \quad \text{for} \quad \boldsymbol{x}=(x, y, z) \neq \boldsymbol{\xi} \tag{2.10}$$

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|} + \text{regular function (in the neighbourhood of } \boldsymbol{x}=\boldsymbol{\xi}) \tag{2.11}$$

$$G=0 \quad \text{on } y=0\,, \text{ say } S^* \tag{2.12}$$

$$\frac{\partial G}{\partial n} = 0 \text{ on } f(\boldsymbol{x})=0\,, \text{ say } S^0 \tag{2.13}$$

$$G \to 0 \quad \text{as} \quad |\boldsymbol{x}| \to \infty\,. \tag{2.14}$$

Application of Green's theorem to the fluid region leads to the following explicit expression for $\psi_i(\boldsymbol{x})$. It follows that

$$4\pi\psi_i(\boldsymbol{x}) = \left(\iint_{S^0} + \iint_{S^*}\right)\left(G\frac{\partial\psi_i}{\partial n} - \psi_i\frac{\partial G}{\partial n}\right)d\sigma \tag{2.15}$$

and with the boundary conditions for G and ψ_i the following results are derived

$$4\pi\psi_0(\boldsymbol{x}) = \iint_{S^0} G\boldsymbol{U}\cdot\nabla f d\sigma \tag{2.16}$$

and

$$4\pi\psi_i(\boldsymbol{x}) = -\iint_{S^*} \frac{\partial\psi_{i-1}}{\partial n}(\xi, 0, \zeta)\frac{\partial G}{\partial n}(\boldsymbol{x}, \xi, 0, \zeta)d\xi d\zeta \text{ for } i=1, 2, \ldots \tag{2.17}$$

Although (2.16) and (2.17) represent an explicit expression for $\psi_i(\boldsymbol{x})$ it is not an easy task to determine the appropriate Green's function in this general case. Once this Green's function has been found the regular solution is known.

Some remarks will be made about the construction of the Green's function. An examination of condition (2.12) leads to the conclusion that the problem is antisymmetric in y. $G(\boldsymbol{x}, \boldsymbol{\xi})$ can be written as follows

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|} - \frac{1}{|\boldsymbol{x}^*-\boldsymbol{\xi}|} + g(\boldsymbol{x}, \boldsymbol{\xi})\,, \text{ where } \boldsymbol{x}^*=(x, -y, z) \tag{2.18}$$

where $g(\boldsymbol{x}, \boldsymbol{\xi})$ represents a regular function which is antisymmetric in y. To determine $g(\boldsymbol{x}, \boldsymbol{\xi})$ a rather complicated integral equation has to be solved. This can be done numerically and is not considered in this paper.

Because of the behaviour of $f_y(x, y, z)$ near $y=0$ it is easy to verify that ψ_0 in (2.16) is a regular function near $y=0$ with $\psi_0(x, 0, z)=0$ and $\psi_{0_y}(x, 0, z)=V(x, z)$. This does not hold when horizontal motions of the object are prescribed. In that case the vertical velocity becomes singular near the object and the free surface. In the problem we are dealing with it turns out that singularities similar to the horizontal motion solution play a role in the higher order approximations of the heave problem. For the determination of the wave contribution the singularities play a role of minor importance in both cases. In the two-dimensional problem we came across similar behaviour [1].

From now on we consider the vertical fluid velocity near the object and the free surface to be known and given by

$$\psi_{0_y}(x, 0, z) = V(x, z) \quad \text{for} \quad f(x, 0, z) \simeq 0\,, \tag{2.19}$$

to the lowest order of approximation. If the object is a semi-submerged sphere this velocity can be calculated exactly with the help of the known Green's function. We then get

$$\psi_0(x, y, z) = -\tfrac{1}{2}U\frac{\rho_0^3}{R^3}y \tag{2.20}$$

for the geometrical configuration

$$f(x, y, z) = x^2 + y^2 + z^2 - \rho_0^2 = R^2 - \rho_0^2 = 0 .$$

Hence in the spherical case

$$\psi_{0y}(x, 0, z) = V(x, z) = -\tfrac{1}{2}U \text{ for } x^2 + z^2 = \rho_0^2 . \tag{2.21}$$

With the help of (2.19) the wave solution will be calculated in the next section.

3. The inner solution

In this section a discussion of the local solution will be presented. Because of the reduction of the surface condition to a condition without a derivative in it, the wave contribution could not be found in the last section. In order to obtain this wave part we have to stretch the y-coordinate. When we introduce $y' = ky$ both terms in the complete free surface become of the same order, however, because of condition (2.3) a local solution can be obtained of the form

$$\Phi(x, y, z) = \chi(x, z)\, e^{ky} \tag{3.1}$$

where $\chi(x, z)$ is a solution of the two-dimensional Helmholtz equation

$$\chi_{xx} + \chi_{zz} + k^2 \chi = 0 \tag{3.2}$$

satisfying the boundary condition at $f(x, y, z) = 0$ and the radiation condition at infinity. The condition at $f(x, y, z)$ cannot be met, because of the exponential behaviour with respect to y in (3.1) while (2.4) does not admit such behaviour. Therefore an inner solution of the local solution has to be considered which should be matched with an outer local solution of the form (3.1).

For the construction of the inner solution we may stretch the coordinates in the neighbourhood of S^+ and a point $(x_0, 0, z_0)$ of S^0 as follows:

$$x' = k(x - x_0(\sigma)) , \; y' = ky , \; z' = k(z - z_0(\sigma)) , \tag{3.3}$$

where the relation $f(x_0(\sigma), 0, z_0(\sigma)) = 0$ holds, and where the intersection of the object with the free surface is given in parametric form by:

$$\boldsymbol{x}_0(\sigma) = (x_0(\sigma), 0, z_0(\sigma)) . \tag{3.4}$$

A more suitable coordinate system can be found by introducing the distance τ along the outward normal to (3.4) in the plane $y = 0$, and the arclength σ along the intersection curve. Hence the coordinates are now (σ, y, τ) and the arclength ds in this new system is equal to

$$(ds)^2 = (d\sigma)^2 + \left\{ \frac{\rho(\sigma) + \tau}{\rho(\sigma)} \right\}^2 (d\tau)^2 + (dy)^2 \tag{3.5}$$

where ρ is the signed radius of curvature (because $\tau > 0$ in the outward direction). The transformation formulae are (see figure2)

$$x = x_0(\sigma) + \tau\rho \frac{d^2 x_0(\sigma)}{d\sigma^2} \quad y = y , \quad z = z_0(\sigma) + \tau\rho \frac{d^2 z_0(\sigma)}{d\sigma^2} . \tag{3.6}$$

After substitution of (3.6) (σ, y, τ) are stretched

$$y' = ky , \quad \tau' = k\tau , \quad \sigma' = \sigma . \tag{3.7}$$

The differential equation for Φ becomes

$$\Delta\Phi = \frac{\partial^2 \Phi}{\partial \sigma^2} + \frac{\rho^2}{(\rho+\tau)^2} \frac{\partial^2 \Phi}{\partial \tau^2} + \frac{\partial^2 \Phi}{\partial y^2} - \frac{\tau\rho'}{\rho(\rho+\tau)} \frac{\partial \Phi}{\partial \sigma} - \frac{\rho^2}{(\rho+\tau)^3} \frac{\partial \Phi}{\partial \tau} =$$

$$= k^2 \left(\frac{\partial^2 \Phi}{\partial \tau'^2} + \frac{\partial^2 \Phi}{\partial y'^2} \right) - O(k) = 0 , \tag{3.8}$$

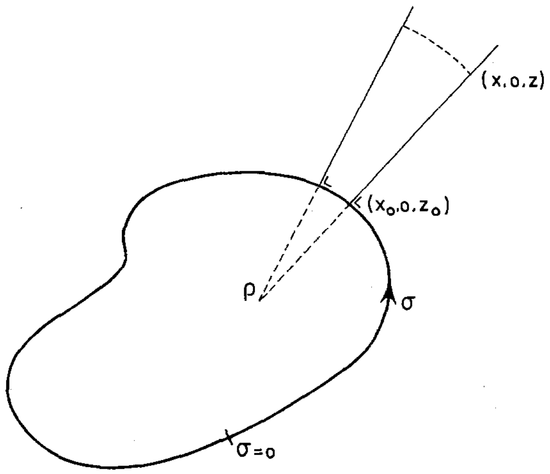

Figure 2

with free surface condition:

$$\frac{\partial \Phi}{\partial y'} - \Phi = 0 \text{ at } y' = 0 .$$

The boundary condition at S^0 needs some further investigation. The equation of the surface S^0 becomes

$$f(x, y, z) = f^*(\sigma, \tau, y) = f^*\left(\sigma, \frac{\tau'}{k}, \frac{y'}{k}\right) =$$

$$= f^*(\sigma, 0, 0) + \frac{\tau'}{k} f_\tau^* + \frac{y'}{k} f_y^* + \frac{(\tau)^2}{2k^2} f_{\tau\tau}^* + \frac{\tau' y'}{k^2} f_{\tau y}^* + \frac{(y')^2}{2k^2} f_{yy}^* + O\left(\frac{1}{k^3}\right) = 0 \tag{3.9}$$

After some calculation it follows that this is approximately equivalent with:

$$\tau' \simeq -\frac{(y')^2}{2ka}, \quad \text{where } a = f_\tau^* / f_{yy}^* = O(1) . \tag{3.10}$$

Thus the boundary condition at S^0 becomes

$$\frac{\partial \Phi}{\partial n} = k \left\{ \frac{\partial \Phi}{\partial \tau'} + \frac{y'}{ka} \frac{\partial \Phi}{\partial y'} + \ldots \right\} =$$

$$= U \cdot \nabla f \simeq \frac{Uy}{a} = \frac{Uy'}{ka} \text{ at } \tau' \simeq -\frac{(y')^2}{2ka} .$$

This yields the condition

$$\frac{\partial \Phi}{\partial \tau'} + \frac{y'}{ka} \frac{\partial \Phi}{\partial y'} \simeq \frac{Uay'}{(ka)^2} \text{ at } \tau' \simeq -\frac{(y')^2}{2ka} .$$

Furthermore for the wave solution the outgoing condition has to be satisfied and the total solution has to match with the regular solution for $y \to -\infty$. To avoid this matching procedure another approach can be followed which leads to the lowest order approximation with less effort. The method is described by R. E. O'Malley Jr. [5] for a system of first order ordinary differential equations. The crucial step is to write the solution as a superposition of the regular solution and the local solution. This solution is valid in the fluid domain near the object.

We write

$$\Phi \sim \sum_{i=0}^{\infty} k^{-i} \psi_i(x, y, z) + \sum_{i=0}^{\infty} \Phi_i(\sigma, \tau', y') \tag{3.11}$$

where $\Phi_i = o(\Phi_{i-1})$ for $k \gg 1$ and ψ_i is defined in section 2. Some calculations show that $\partial\psi_0/\partial y'$ leads to a contribution to the boundary condition at S^0 while the contribution of $\partial\psi_0/\partial\tau'$ is of lower order in k. Hence Φ_0 has to be a solution of

$$\frac{\partial^2 \Phi_0}{\partial \tau'^2} + \frac{\partial^2 \Phi_0}{\partial y'^2} = 0 \,, \quad \text{in the fluid} \,, \tag{3.12}$$

$$\frac{\partial \Phi_0}{\partial \tau'} = \frac{a(U - V(\sigma))}{(ka)^2} \quad y' = \frac{W(\sigma)}{(ka)^2} y' \,, \text{ at } \tau' = 0 \,, \tag{3.13}$$

$$\frac{\partial \Phi_0}{\partial y'} - \Phi_0 = 0 \,, \text{ on } y' = 0 \,, \tag{3.14}$$

$$\Phi_0 \to 0 \text{ as } y' \to -\infty \,,$$

Furthermore the outgoing condition (for the waves) has to be satisfied. In the special case of a semi-submerged sphere $W(\sigma) \equiv \frac{3}{2}\rho_0 U$.

The problem is reduced to the two-dimensional problem [1]. The wave part of the solution can be found easily. For the construction of Φ_0 an appropriate Green's function must be found. The Green's function $g(\tau', y'; \xi, \eta)$ is a solution of

$$\frac{\partial^2 g}{\partial \tau'^2} + \frac{\partial^2 g}{\partial y'^2} = 0 \text{ for } (\tau', y') \neq (\xi, \eta) \text{ say } \tau \neq \xi \,,$$

$$\frac{\partial g}{\partial y'} - g = 0 \qquad \text{at } y' = 0 \,,$$

$$\frac{\partial g}{\partial \tau'} = 0 \qquad \text{at } \tau' = 0 \,,$$

$$g(\tau, \xi) = \ln|\tau - \xi| + (\text{regular function}) \,.$$

The Green's function has to meet the outgoing condition as well.

First the condition at $\tau' = 0$ will be disregarded. Later on this condition will be met by means of reflexion as follows

$$\begin{aligned} g(\tau', y'; \xi, \eta) &= g^0(\tau', y'; \xi, \eta) + g^0(-\tau', y'; \xi, \eta) \\ &= g^0(\tau', y'; \xi, \eta) + g^0(\tau', y'; -\xi, \eta) \,. \end{aligned}$$

The function $g^0(\tau, \xi)$ is well known (John [2]) and may be written in the form

$$g^0(\tau', y'; \xi, \eta) = -2\pi i \exp\{i|\tau' - \xi| + (y' + \eta)\} + \tfrac{1}{2} \ln\left\{\frac{(\tau' - \xi)^2 + (y' - \eta)^2}{(\tau' - \xi)^2 + (y' + \eta)^2}\right\} +$$
$$-2 \int_0^\infty \frac{t \cos(y' + \eta)t + \sin(y' + \eta)t \, e^{-|\tau' - \xi|t}}{1 + t^2} dt \,. \tag{3.15}$$

It can be shown that for large values of τ' or ξ, g^0 behaves like

$$g^0(\tau', y'; \xi, \eta) \sim -2\pi i \exp\{i|\tau' - \xi| + (y' + \eta)\} + O\left(\frac{1}{\tau' - \xi}\right) . \tag{3.16}$$

We now apply Green's theorem as explained in section 2

$$\Phi_0(\tau', y') = \frac{2W(\sigma)}{(ka)^2 \, 2\pi} \int_{-\infty}^{0} \eta g^0(\tau', y'; 0, \eta) d\eta \,.$$

We have to deal with the finite part of this integral, because the wave solution has a meaning only for large values of τ'.

The wave part (for large values of τ') is of the form

$$\Phi_0 = \frac{-2i W(\sigma)}{(ka)^2} e^{i\tau' + y'} \int_{-\infty}^{0} \eta \, e^{\eta} \, d\eta = \frac{2i W(\sigma)}{(ka)^2} e^{i\tau' + y'} \,. \tag{3.67}$$

This is the inner wave solution of the local solution. This solution provides the necessary matching condition for the outer solution.

4. The outer solution

In section 3 a suitable form has been introduced for the local solution (3.1) at finite distance from the S^0 and near the free surface. In this section we apply the ray method to obtain the wave solution in the far field. Hence, we introduce

$$\Phi(x, y, z) \sim e^{ik\, S(x,z)+ky} \sum_{n=0}^{\infty} (ik)^{-n} v_n(x,z) . \tag{4.1}$$

Clearly condition (2.3) is met by (4.1). Equation (2.2) produces after substitution of (4.1) the well known eiconal and transport equations for the unknown functions.

The eiconal equation turns out to be

$$(\nabla S)^2 = S_x^2 + S_z^2 = 1 \tag{4.2}$$

and the transport equations are

$$2\nabla S \cdot \nabla v_m + v_m \nabla S = -\Delta v_{m-1} \text{ for } m = 0, 1, 2, \ldots ; v_{-1} = 0 \tag{4.3}$$

The boundary condition for $S(x, z)$ results from (2.4). It follows that $S(x_0(\sigma), z_0(\sigma))=0$ where $\boldsymbol{x}_0(\sigma)=(x_0(\sigma), z_0(\sigma))$ represents the intersection curve with the free surface. $S(x, z)$ becomes uniquely determined if the outgoing condition is fulfilled.

The characteristic curves (in this theory called rays) are orthogonal to the wave fronts S = constant. Hence, the rays are orthogonal to the curve $\boldsymbol{x}_0(\sigma)=(x_0(\sigma), z_0(\sigma))$ because $S\equiv 0$ on this curve. Furthermore, since the right-hand side of the eiconal equation equals a constant, the rays are straight lines given in (3.6).

The function S becomes

$$S(x, z) = \tau \tag{4.4}$$

This function S is a solution of the eiconal equation and matches exactly the phase function of the inner solution. The transport equation for $v_0(x, z)$ is easily solved as well:

$$v_0(\sigma, \tau) = v_0(\sigma, 0)\left(\frac{\rho}{\rho+\tau}\right)^{\frac{1}{2}} \tag{4.5}$$

The function $v_0(\sigma, 0)$ which appears as an integration constant follows after matching the outer

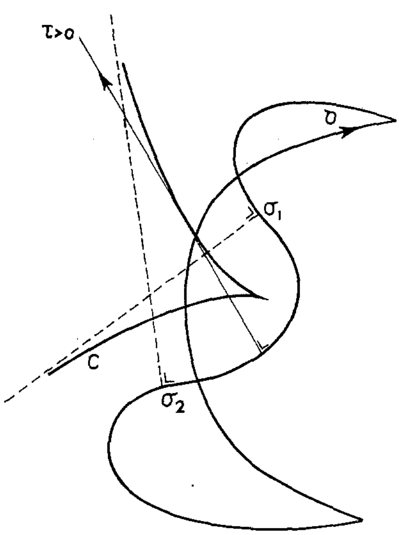

Figure 3

solution with the inner solution. The matching rule

$$(v_{0\,\text{inner}})_{\text{outer}} \equiv (v_{0\,\text{outer}})_{\text{inner}}$$

leads to

$$v_0(\sigma, 0) = \frac{2\mathrm{i}\,W(\sigma)}{(ka)^2}.$$

The outer solution of the local solution equals

$$\Phi(x, y, z) \simeq \frac{2\mathrm{i}\,W(\sigma)}{(ka)^2}\left(\frac{\rho}{\rho+\tau}\right)^{\frac{1}{2}} \mathrm{e}^{ik\tau+ky} \tag{4.6}$$

This solution is uniformly valid provided that $\tau = -\rho$ lies inside the object. This is the case if $\boldsymbol{x}_0(\sigma) = (x_0(\sigma), z_0(\sigma))$ is a convex curve. If several rays pass through a point, the final solution is a superposition of all ray contributions. At one side of a caustic always two "neighboring" rays pass through a point; if we are at the other side no such rays are found. This does not exclude the presence of any rays in that region. This can be seen in fig. 3.

5. The uniform solution

In this section we consider a non-convex curve as the intersection of S^0 and S^+. There exists a range of σ, say $\sigma_1 < \sigma < \sigma_2$ where $\rho+\tau$ becomes zero. This is possible because ρ is negative with our definition of the positive direction along the normal (fig. 3).

Evidently (4.6) becomes infinite near the caustic C. Ludwig [4] suggests the application of another "Ansatz" than the one used in section 4. In the neighbourhood of a caustic the solution can be written as a combination of an Airy function and its derivative as follows

$$\Phi(x,y,z) = \mathrm{e}^{ik\,\theta(\boldsymbol{x})+ky}\left\{\mathrm{Ai}(-k^{\frac{2}{3}}\rho(\boldsymbol{x}))g(\boldsymbol{x},k) + \frac{i}{k^{\frac{1}{3}}}\mathrm{Ai}'(-k^{\frac{2}{3}}\rho(\boldsymbol{x}))h(\boldsymbol{x},k)\right\} \tag{5.1}$$

where $\boldsymbol{x} = (x, z)$ and the amplitude functions g and h can be written as power series with respect to k^{-i}. We only take the lowest order approximation with index zero into account. Here $\mathrm{Ai}(w)$ is the Airy function satisfying the differential equation

$$\mathrm{Ai}''(w) - w\,\mathrm{Ai}(w) = 0\,. \tag{5.2}$$

If (5.1) is substituted into (2.2) and (5.2) is employed, the result is

$$\begin{aligned} \mathrm{e}^{ik\,\theta(\boldsymbol{x})+ky}\Big\{ & -k^2\mathrm{Ai}(-k^{\frac{2}{3}}\rho)\,g_0[(\nabla\theta)^2+\rho(\nabla\rho)^2-1] \\ & + k^{\frac{4}{3}}\mathrm{Ai}'(-k^{\frac{2}{3}}\rho)\,h_0[(\nabla\theta)^2+\rho(\nabla\rho)^2-1] \\ & - ik^{\frac{5}{3}}\mathrm{Ai}'(-k^{\frac{2}{3}}\rho)\,g_0[2\nabla\theta\cdot\nabla\rho] + k^{\frac{5}{3}}\mathrm{Ai}''(-k^{\frac{2}{3}}\rho)\,h_0[2\nabla\theta\cdot\nabla\rho] \\ & + ik\,\mathrm{Ai}(-k^{\frac{2}{3}}\rho)[2\nabla\theta\cdot\nabla g_0 + \Delta\theta g_0 + 2\rho\nabla\rho\cdot\nabla h_0 + \rho\Delta\rho h_0 + (\nabla\rho)^2 h_0] \\ & - k^{\frac{2}{3}}\mathrm{Ai}'(-k^{\frac{2}{3}}\rho)[2\nabla\rho\cdot\nabla g_0 + \Delta\rho g_0 + 2\nabla\theta\cdot\nabla h_0 + \Delta\theta h_0] \\ & + \mathrm{Ai}(-k^{\frac{2}{3}}\rho)\Delta g_0 + \mathrm{i}\frac{\mathrm{Ai}'(-k^{\frac{2}{3}}\rho)}{k^{\frac{1}{3}}}\Delta h_0 \Big\} = 0\,. \end{aligned} \tag{5.3}$$

Each term in square brackets is separately equated to zero. The first four terms lead to the equations

$$(\nabla\theta)^2 + \rho(\nabla\rho)^2 - 1 = 0 \tag{5.4}$$

$$2\nabla\theta\cdot\nabla\rho = 0\,. \tag{5.5}$$

First we make a connection between (5.4)–(5.5) and the eiconal equation. Consider the region $\rho > 0$, i.e. the illuminated region as shown in figure 4.

In this region the nonlinear system of partial differential equations (5.4)–(5.5) is hyperbolic. Multiplying (5.5) by $\pm\rho^{\frac{1}{2}}$ and adding (5.4), we obtain

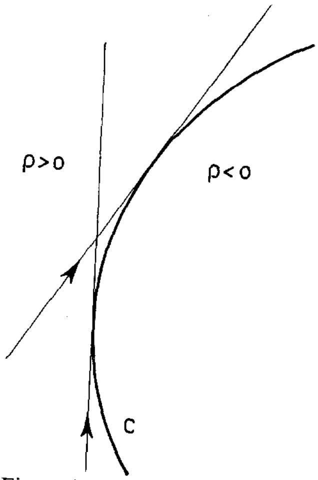

Figure 4.

$$(\nabla\theta \pm \rho^{\frac{1}{2}} \nabla\rho)^2 = 1 \,. \tag{5.6}$$

Introducing

$$S^{\pm} = \nabla\theta \pm \rho^{\frac{1}{2}} \nabla\rho \tag{5.7}$$

it follows that $S^{\pm}$ are the phase functions constructed in section 4, which are known. Hence, θ and ρ are known and given by

$$\theta = \tfrac{1}{2}(S^{+} + S^{-}) \,, \; \tfrac{2}{3}\rho^{\frac{3}{2}} = \tfrac{1}{2}(S^{+} - S^{-}) \,. \tag{5.8}$$

It can be shown that both θ and ρ are regular functions. For $\rho < 0$ it can be shown that (5.1) becomes exponentially small for large k. For $\rho > 0$ the asymptotic values of the phase equals the values found by means of the ray method.

The next pair of terms between square brackets in equation (5.3) lead to the equations

$$2\nabla\theta \cdot \nabla g_0 + \Delta\theta g_0 + 2\rho \nabla\rho \cdot \nabla h_0 + \rho \Delta\rho \, h_0 + (\nabla\rho)^2 h_0 = 0 \tag{5.9}$$

$$2\nabla\rho \cdot \nabla g_0 + \Delta\rho \, g_0 + 2\nabla\theta \cdot \nabla h_0 + \Delta\theta h_0 = 0 \,. \tag{5.10}$$

The equations (5.9) and (5.10) are connected with the transport equation. Multiplying (5.10) by $\pm\rho^{\frac{1}{2}}$ and adding (5.9) we obtain

$$2(\nabla\theta \pm \rho^{\frac{1}{2}} \nabla\rho) \cdot \nabla(g_0 \pm \rho^{\frac{1}{2}} h_0) + (\Delta\theta \pm \rho^{\frac{1}{2}} \Delta\rho) \cdot (g_0 \pm \rho^{\frac{1}{2}} h_0) = 0 \,.$$

Introducing $G^{\pm} = g_0 \pm \rho^{\frac{1}{2}} h_0$, we obtain the equation

$$2\nabla S^{\pm} \cdot \nabla G^{\pm} + [\Delta S^{\pm} \mp \tfrac{1}{2}\rho^{-\frac{1}{2}} (\nabla\rho)^2] G^{\pm} = 0$$

The singularity in the last part of the term between square brackets cancels the singularity of the first part.

The substitution $v_0^{\pm} = \rho^{-\frac{1}{4}} G^{\pm}$ leads to the transport equation of section 4

$$2\nabla S^{\pm} \cdot \nabla v_0^{\pm} + \Delta S^{\pm} v_0^{\pm} = 0$$

The conclusion is that if the multiple ray solution is found at finite distance from the caustic, the uniform asymptotic solution is calculated easily.

REFERENCES

[1] A. J. Hermans, A perturbation method for the radiation of surface waves, *J. of Eng. Math.*, 6, 4 (1972) 323–330.
[2] F. John, On the motion of floating bodies II, *C.P.A.M.*, 3 (1950) 45–103.
[3] J. B. Keller and R. M. Lewis, Asymptotic theory of wave propagation and diffraction, Book, to appear.
[4] D. Ludwig, Uniform asymptotic expansions at a caustic, *C.P.A.M.*, 19 (1966) 215–250.
[5] R. E. O'Malley, Jr., Boundary layer methods for nonlinear initial value problems, *SIAM Rev.*, 13 (1971) 425–433.
[6] F. Ursell, Short surface waves due to an oscillating immersed body, *Proc. of the Camb. Phil. Soc. A* 220 (1950) 90–103.